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Changes in the Effective Parameters of Averaged Motion in Nonlinear Systems Subject to Noise

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We discuss how the effective parameters characterising averaged motion in nonlinear systems are affected by noise (random fluctuations). In this approach to stochastic dynamics, the stochastic system is replaced by its deterministic equivalent but with noise-dependent parameters. We show that it can help to resolve certain paradoxes and that it has a utility extending far beyond its usual application in passing from the microscopic equations of motion to the macroscopic ones. As illustrative examples, we consider the diode-capacitor circuit, a Brownian ratchet, and a generic stochastic resonance system. In the latter two cases we calculate for the first time their effective parameters of averaged motion as functions of noise intensity. We speculate that many other stochastic problems can be treated in a similar way.

KEY WORDS: noise, nonlinear dynamics, fluctuational phenomena, Brownian motion
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1. INTRODUCTION

Consideration of fluctuation-induced changes in the effective parameters of averaged motion can provide an effective way of calculating the effect of noise on a nonlinear system. In what follows, we obtain noise-dependent effective parameters for two problems of topical interest: Brownian ratchets and stochastic resonance. The underlying idea is novel in the sense that physicists do not normally approach noisy systems in this way. At the same time, however, there is a sense in which the idea is a very old one. As a classical example, we note that the Stokes force in the

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Langevin equation for a Brownian particle[(1)] arises from noise-induced changes in the effective parameters of averaged motion: this force is caused by collisions of the Brownian particle with the molecules of the surrounding gas, and these collisions can of course be considered as noise.

This approach lies at the root of statistical mechanics, which is the only place where physicists normally encounter it. The central result is that the equations of averaged motion are irreversible, whereas the initial equations describing motion at the molecular level are reversible (Hamiltonian). The main contribution to the solution of this problem was by Boltzmann[(2)] who showed how to obtain equations describing irreversible processes starting from reversible Hamiltonian mechanics. However, the Boltzmann theory is valid only for ideal gas, because he took into account only paired molecular collisions. As a result, his equation describes the effect of the molecular collisions only on the dissipative processes responsible for the establishment of equilibrium state, but not on averaged characteristics such that pressure, internal energy and so on. As distinct from the Boltzmann equation, the Bogolyubov method allows one to do this, at least in principle. The problem was solved more exactly by Klimontovich[(1)] and later explored through analogue simulation[(3)] using electronic circuit models.[(4)]

One can readily identify a number of cases where changes in the effective parameters of averaged motion manifest in macroscopic physics; the concept is *not* restricted to its conventional application in passing from the microscopic equations to the macroscopic ones. As one example, we mention turbulent jets. It is known that within the initial part of the jet the thickness of the boundary layer increases with increasing distance from the nozzle.[(5)] One of the first theories explaining this fact was due to Prandtl,[(6)] who introduced the notion of the turbulent viscosity ν_{turb} caused by turbulent pulsations, here regarded as noise. It depends on the distance from the nozzle and can be written approximately as

$$\nu_{\text{turb}} = a\delta(x)\Delta U,$$

where $\delta(x)$ is the boundary layer thickness, ΔU is the velocity difference in the boundary layer region, and a is a coefficient of proportionality. Many results obtained by use of this formula gave good agreement with experiments. Landau noted,[(7)] however, that due to the presence of arbitrary parameters, e.g. a, other expressions for the turbulent viscosity can also give good agreement with experiment which is why in our own earlier work[(5)] on jet turbulence we did not use the notion of turbulent viscosity. However, from our theoretical results we can estimate the ratio between the turbulent (ν_{turb}) and kinematic (ν) viscosities. For a Reynolds number of 25000, at the end of the jet's initial part ν_{turb}/ν was found to be $\sim$100 which agrees in order of magnitude with Landau's formula $\nu_{\text{turb}} \sim \nu \mathsf{Re}/\mathsf{Re}_{\text{cr}}$, where Re_{cr} is the critical Reynolds number.[(7)] For submerged jets $\mathsf{Re}_{\text{cr}} \sim 100$.

We mention also a paradox related to turbulence. First described by Prandtl,[8] and then studied by Ginevsky and Kolesnikov,[9] it takes the following form. Numerous observations show that a barge travelling with the stream of a river passes ahead of the stream to such an extent that it may be steered with a rudder. Prandtl's explanation of this paradox relies on the formation of a turbulent boundary layer around the barge, possessing a smaller turbulent viscosity than surrounding water and, therefore, exhibiting a smaller resistance to barge motion. Ginevsky and Kolesnikov calculated the magnitude of this effect and showed that the barge velocity should be about half as much again as the stream velocity. So, the resolution of this seeming paradox lies in the strong effect of random turbulent pulsations on the resistance force.

Another problem where changes in the effective parameters of averaged motion in systems with noise play a crucial role is a seeming paradox related to the possibility of realizing Maxwell's demon in an electrical circuit. During the 1950s a great debate took place, involving many physics journals,[10–14] aimed at resolving an apparent paradox associated with the simplest electrical rectifier circuit (Fig. 1) consisting of a capacitor and diode. It was shown[11,12] that in such a circuit the capacitor can be charged without an external source, only at the expense of thermal fluctuations, i.e. the diode plays the role of Maxwell's demon. This paradoxical result seemed to cast doubt on the validity of the Second Law of Thermodynamics as applied to the phenomenon considered.[14] As far back as 1950, Brillouin[10] showed, by considering the diode as a nonlinear resistor, that validity of the Second Law would require a shift of the voltage-current characteristic of the nonlinear resistor. Stratonovich[15] established that, for a certain model of the diode, such a shift does indeed occur and he calculated its magnitude. Consequently, for the particular case of thermal fluctuations, the mean value of the voltage drops across the capacitor and the mean current in the circuit vanishes. The resolution of this paradox may be considered as a concrete example of passing from the microscopic equations to the macroscopic ones in the equilibrium case, resulting in a change of

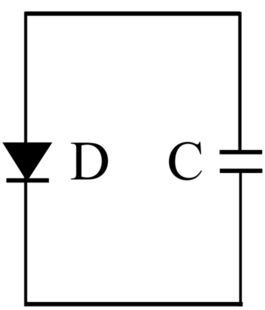

Fig. 1. Schematic image of an electrical rectifier.

the voltage-current diode characteristic. Note that a very similar problem, albeit for a mechanical rectifier (ratchet and pawl) was solved by Feynman.[16]

Finally, we mention a third problem where noise-induced changes of the effective parameters of averaged motion are of fundamental importance, that of stochastic resonance (SR) where a weak periodic signal in a nonlinear system can be optimally amplified by the addition of noise of the appropriate intensity. In the high-noise, weak-signal limit, SR is well-approximated by linear response theory, using a susceptibility derivable from the fluctuation dissipation theorem.[17] Many authors,[18,19] however, have treated SR as a coincidence of the signal period $T = 2\pi/\omega$ and doubled mean first passage time through the potential barrier U_0

$$T_{tr} = \frac{\pi}{\sqrt{2}} \exp\left(\frac{2U_0}{K}\right).$$

Three concerns may be raised in relation to the latter treatment: (i) resonance as conventionally understood cannot occur in a system with only half a degree of freedom, such as the overdamped oscillator usually considered; (ii) if the origin of SR lies in the coincidence of some frequencies, than it should not be of any importance how this coincident frequency is varied, but in an overdamped oscillator the maximum of the signal, although a function of frequency, is not of a resonant character (the response/drive ratio being a monotonically decreasing function of frequency; and (iii) the variance of the jump frequency is very large, the mean square value of the jump frequency is approximately equal to twice its mean. For these reasons, Landa suggested an alternative approach, treating SR as a noise-induced change in the system's effective parameters.[20,21] It facilitates the creation of a rigorous theory of SR, based on the single assumption that the signal is small.

There are numerous other cases of phenomena that can be well-described in terms of noise-induced changes in effective parameters. For example, the possibility of noise-induced directed movement of a particle in a system without any directional forces is readily be accounted for, as we shall see. This problem may of course be solved by other ways too, e.g. by direct solution of the corresponding Fokker–Planck equation. But its consideration from the viewpoint of the noise-induced change of the force acting on the particle gives a better insight into mechanism of this movement.

It should be noted that, in some respects, the problem under consideration is similar to a phenomenon well-known in mechanics where the effective parameters of slow motion change under the action of high-frequency vibrations.[22] We note also that one aspect of this problem was described in the Russian literature many years ago[23,24] through consideration of the effect of Gaussian noise on different nonlinear elements. The authors applied the method of equivalent linearization,[25,26] thereby finding the effective characteristics of these elements.

In reviewing the resolution of the diode-capacitor paradox in Sec. 2, we follow the argument introduced by Stratonovich.[15] We consider a Brownian ratchet in Sec. 3 and SR in Sec. 4, in each case calculating for the first time the noise-dependence of their effective parameters. Finally, in Sec. 5, we point out that the phenomena observed in each of these diverse physical systems can be well-described in terms of noise-induced changes in their parameters of averaged motion. It is likely that the same type of approach will be widely applicable.

2. RESOLUTION OF A PARADOX ASSOCIATED WITH AN APPARENT VIOLATION OF THE SECOND LAW OF THERMODYNAMICS

Let us consider the simplest relevant circuit composed just of a diode D and capacitor C (Fig. 1). The voltage drop V across the capacitor is

$$C\dot{V} + I(V) = 0, \tag{1}$$

where $I(V)$ is the current flowing through the diode, and C is the total capacitance of the diode and capacitor.

It is well known that, in thermal equilibrium, the probability distribution for V is[27]

$$w(V) = \sqrt{\frac{C}{2\pi kT}}\,\exp\left(-\frac{CV^2}{2kT}\right), \tag{2}$$

where k is Boltzmann's constant and T is the temperature.

To find $I(V)$, we will assume the diode to be thermionic with two plane-parallel electrodes: cathode and anode. It is known that in the space between the electrodes there is a constant negative spacecharge generating an electric field characterized by the potential $U(x)$, where x is the distance from the cathode. The qualitative dependence of the function $-eU(x)$ on x, where $-e$ is the electronic charge, is illustrated in Fig. 2. We see that in the path of the electrons there is a potential barrier of height equal to the maximum of $-U(x)$, i.e. to $-U(x_0)$, where x_0 is the extreme point.

We assume the velocity distribution of the electrons to be Maxwellian. Only those electrons whose energy exceeds $-eU(x)$ can overcome the potential barrier.

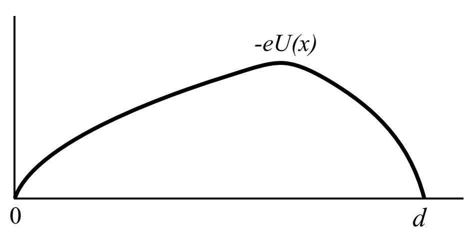

Fig. 2. Qualitative behavior of the function $-eU(x)$, where $-e$ is the charge of an electron, in the absence of the potential difference between the electrodes.

The other electrons return to the cathode. The fraction of electrons overcoming the potential barrier $\propto \exp(-eU(x_0)/(kT))$. It follows that, in the absence of the potential difference, the currents flowing through the diode in each direction are the same, and equal to

$$I_+ = I_- = I_0 \exp\left(-\frac{eU(x_0)}{kT}\right), \tag{3}$$

where I_0 is a constant with the dimensions of current.

If the potential difference between the electrodes is nonzero, then it must vary with time owing to changes in the induced charge on the electrodes. For example, if an electron has left the cathode and is at the point x (where $0 < x < d$, and d is the distance between the electrodes), then the charge on the anode and the potential difference are

$$Q_1(x) = Q - \frac{ex}{d}, \quad V_1(x) = V - \frac{ex}{Cd}, \tag{4}$$

where $V = Q/C$.

When an electron moving in the opposite direction reaches the same point x, the charge on the anode and the potential difference are

$$Q_2(x) = Q + \frac{e(d-x)}{d}, \quad V_2(x) = V + \frac{e(d-x)}{Cd}. \tag{5}$$

It is evident that in the first case the force acting on the electron is

$$F_1(x) = e\,\frac{\partial U(x)}{\partial x} + \frac{eV}{d} - \frac{e^2 x}{Cd^2}, \tag{6}$$

whereas in the second case it is

$$F_2(x) = e\,\frac{\partial U(x)}{\partial x} + \frac{eV}{d} + \frac{e^2(d-x)}{Cd^2}. \tag{7}$$

The extreme points x_1 and x_2 can be found from the equations

$$F_1(x_1) = 0, \quad F_2(x_2) = 0. \tag{8}$$

Noting that $U(0) = U(d) = 0$ and integrating (6) over x from $x = 0$ to $x = x_1$ and (7) from $x = d$ to $x = x_2$, we can calculate the height of the potential barrier in each case:

$$\begin{aligned} U_1 &= -\frac{1}{e}\int_0^{x_1} F_1(x)\,dx = -U(x_1) - V\,\frac{x_1}{d} + \frac{e^2 x_1^2}{2Cd^2}, \\ U_2 &= -\frac{1}{e}\int_d^{x_2} F_2(x)\,dx = -U(x_2) + V\,\frac{d-x_2}{d} + \frac{e^2(d-x_2)^2}{2Cd^2}. \end{aligned} \tag{9}$$

By analogy with (3) we find

$$I_+ = I_0 \exp\left(-\frac{eU_1}{kT}\right), \quad I_- = I_0 \exp\left(-\frac{eU_2}{kT}\right). \tag{10}$$

The total current is thus equal to

$$I(V) = I_0 \left[\exp\left(-\frac{eU_1}{kT}\right) - \exp\left[-\frac{eU_2}{kT}\right)\right]. \tag{11}$$

Substituting (9) into (11) we find

$$\begin{aligned} I(V) = I_0 \Bigg[& \exp\left(-\frac{eU(x_1)}{kT} + \frac{ex_1}{kTd}V - \frac{e^2x_1^2}{2kTCd^2}\right) \\ & - \exp\left(-\frac{eU(x_2)}{kT} - \frac{e(d-x_2)}{kTd}V - \frac{e^2(d-x_2)^2}{2kTCd^2}\right)\Bigg]. \end{aligned} \tag{12}$$

For simplicity, we assume that the forces F_1 and F_2 are approximately the same and given by

$$F_1 \approx F_2 \approx e\,\frac{\partial U(x)}{\partial x}.$$

In this case x_1 and x_2 are approximately coincident, independent of V and equal to x_0. Hence

$$I(V) \approx I_{00}\left[\exp\left(\frac{ex_0}{dkT}(V - V_{01})\right) - \exp\left(-\frac{e(d-x_0)}{kTd}(V + V_{02})\right)\right], \tag{13}$$

where $I_{00} = I_0 \exp(-eU(x_0)/(kT))$,

$$V_{01} = \frac{ex_0}{2Cd}, \quad \text{and} \quad V_{02} = \frac{e(d-x_0)}{2Cd}. \tag{14}$$

It is seen from (13) that $I(V) = 0$ for

$$V = V_0 = \frac{x_0 V_{01} - (d - x_0)V_{02}}{d}, \tag{15}$$

i.e. the dependence of the current on the potential difference V is displaced from the origin ($I(0) \neq 0$). Only for $x_0 V_{01} = (d - x_0)V_{02}$ is the displacement absent. But in this case the system under consideration would not act as a rectifier.

This displacement of the diode voltage-current characteristic is precisely what results in the vanishing of the mean current flowing through the diode, and of the voltage, as required by the Second Law of Thermodynamics. We can find the actual mean current from (2), which yields:

$$\langle I(V)\rangle = I_{00}\sqrt{\frac{C}{2\pi kT}} \int\limits_{-\infty}^{\infty} \left[\exp\left(\frac{ex_0}{dkT}(V - V_{01})\right)\right.$$

$$- \exp\left(-\frac{e(d-x_0)}{dkT}(V+V_{02})\right)\Bigg] \times \exp\left(-\frac{CV^2}{2kT}\right) dV$$

$$= \exp\left[-\frac{ex_0}{dkT}\left(V_{01}-\frac{ex_0}{2Cd}\right)\right]$$

$$- \exp\left[-\frac{e(d-x_0)}{dkT}\left(V_{02}-\frac{e(d-x_0)}{2Cd}\right)\right]. \tag{16}$$

Using (14) we then find that $\langle I(V)\rangle = 0$, thereby resolving the paradox.

3. TRANSPORT OF A LIGHT PARTICLE IN A VISCOUS MEDIUM WITH A SAW-TOOTH POTENTIAL UNDER THE INFLUENCE OF HARMONIC AND RANDOM FORCES

The phenomenon of noise-induced transport of Brownian particles has attracted considerable interest in recent years, for the most part in the context of biological and chemical problems.[28–35] Consideration is most often restricted to the so-called overdamped case where the motion of a light particle is described by a first order differential equation of the form

$$\dot{x} = -f(x) + \zeta(t) + \xi(t). \tag{17}$$

Here $f(x)$ is a periodic function of x possessing a certain asymmetry, $\zeta(t)$ is a regular or random process, and $\xi(t)$ is white noise modelling thermal fluctuations. One of the simplest forms of $f(x)$ corresponds to the saw-tooth potential $U(x)$ shown in Fig. 3 and is described by

$$f(x) = \begin{cases} +a_1 & \text{for } nL < x < nL + x_1, \\ -a_2 & \text{for } nL - x_2 < x < nL, \end{cases} \tag{18}$$

where $n = 0, \pm 1, \pm 2, \ldots$, and $L = x_1 + x_2$ is the period of the function $f(x)$.

For simplicity we set

$$\zeta(t) = B \sin \omega t. \tag{19}$$

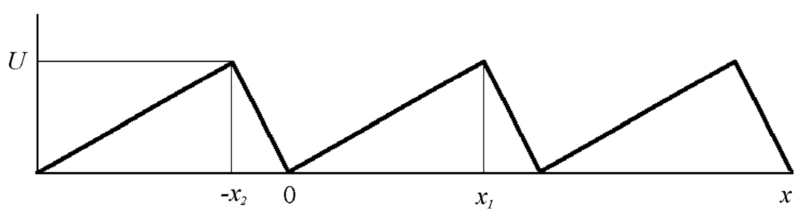

Fig. 3. An example of the saw-tooth potential.

Usually this problem is solved by calculation of the flux of the probability. Here we approach the problem in a different way, by finding the mean value of the force $f(x)$ acting on the particle.

Averaging Eq. (17) over the statistical ensemble and taking account of (19) we obtain an equation for the averaged motion of the particle:

$$v(t) = -\langle f(x)\rangle + B \sin \omega t. \tag{20}$$

where $v(t) = \langle \dot{x} \rangle$ is the averaged particle velocity. Averaging (20) over time we find

$$\overline{v(t)} = -\overline{\langle f(x)\rangle}. \tag{21}$$

To find $v(t)$ and $\langle f(x)\rangle$, we write the Fokker–Planck equation corresponding to the Langevin equation (17):

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left((f(x) - B \sin \omega t) w(x,t) + \frac{K}{2}\frac{\partial w(x,t)}{\partial x}\right). \tag{22}$$

Because $f(x)$ is a periodic function of x, the probability density $w(x,t)$ is also a periodic function of x. Hence Eq. (22) need only be solved in the interval from $-x_2$ to x_1.

Equation (22) can be conveniently rewritten as

$$\frac{\partial w}{\partial t} = -\frac{\partial G(x,t)}{\partial x}, \tag{23}$$

where

$$G(x,t) = -\left(\frac{K}{2}\frac{\partial w(x,t)}{\partial x} + (f(x) - B \sin \omega t) w(x,t)\right) \tag{24}$$

is the probability flux.

Integrating Eq. (24) over x from $-x_2$ to x_1 and taking into account conditions of periodicity and normalization for $w(x,t)$ we find

$$\langle f(x)\rangle = -\int_{-x_2}^{x_1} G(x,t)\,dx + B \sin \omega t. \tag{25}$$

It follows that the averaged force, as distinct from the initial force $f(x)$, contains both constant and periodic components.

To find $G(x,t)$, we need to solve Eq. (22). As pointed out above, most researchers use the so-called *quasistatic approximation* (also known as the *adiabatic approximation*) to solve this equation, i.e. they neglect the term $\partial w/\partial t$; it is valid for low frequencies ω. We will not use this approximation, but we restrict ourselves to small amplitudes B.

If B is sufficiently small we can seek a solution of Eq. (22) in expanded form retaining terms of order up to B^2:

$$w(x,t) = w_0(x) + w_1(x,t)B + w_2(x,t)B^2. \tag{26}$$

Substituting (26) into Eq. (22) we obtain equations for $w_0(x,t)$, $w_1(x,t)$ and $w_2(x,t)$:

$$\frac{K}{2}\frac{dw_0(x)}{dx} + f(x)w_0(x) = -G_0, \tag{27}$$

$$\begin{aligned} &\frac{\partial w_1}{\partial t} - \frac{\partial}{\partial x}\left(f(x)w_1(x,t) + \frac{K}{2}\frac{\partial w_1(x,t)}{\partial x}\right) + \frac{dw_0(x)}{dx}\sin\omega t = 0,\\ &\frac{\partial w_2}{\partial t} - \frac{\partial}{\partial x}\left(f(x)w_2(x,t) + \frac{K}{2}\frac{\partial w_2(x,t)}{\partial x}\right) + \frac{\partial w_1}{\partial x}\sin\omega t = 0, \end{aligned} \tag{28}$$

where G_0 is the probability flux in the zeroth order approximation with respect to B.

Setting $w_1(x,t) = w_{1s}(x)\sin\omega t + w_{1c}(x)\cos\omega t$, $w_2(x,t) = w_{20}(x)$ we obtain the following equations for $w_{1s}(x)$, $w_{1c}(x)$ and $w_{20}(x)$:

$$\frac{K}{2}\frac{d^2w_{1s}(x)}{dx^2} + \frac{df(x)w_{1s}(x)}{dx} + \omega w_{1c}(x) = \frac{dw_0}{dx}, \tag{29}$$

$$\begin{aligned} &\frac{K}{2}\frac{d^2w_{1c}(x)}{dx^2} + \frac{df(x)w_{1c}(x)}{dx} - \omega w_{1s}(x) = 0,\\ &\frac{K}{2}\frac{dw_{20}(x)}{dx} + f(x)w_{20}(x) - \frac{w_{1s}(x)}{2} = -G_2, \end{aligned} \tag{30}$$

where G_2 is the probability flux in the second order approximation with respect to B.

The probability density $w(x,t)$ must satisfy the conditions of continuity for $x = 0$, periodicity, and normalization. These conditions are

$$\begin{aligned} &w_0(0_-) = w_0(0_+), \quad w_0(-x_2) = w_0(x_1), \quad w_{1s}(0_-) = w_{1s}(0_+),\\ &w_{1c}(0_-) = w_{1c}(0_+), \quad w_{1s}(-x_2) = w_{1s}(x_1), \quad w_{1c}(-x_2) = w_{1c}(x_1),\\ &w_{20}(0_-) = w_{20}(0_+), \quad w_{20}(-x_2) = w_{20}(x_1),\\ &\int_{-x_2}^{x_1} w_0(x)\,dx = 1, \quad \int_{-x_2}^{x_1} w_{1s}(x)\,dx = 0,\\ &\int_{-x_2}^{x_1} w_{1c}(x)\,dx = 0, \quad \int_{-x_2}^{x_1} w_{20}(x)\,dx = 0. \end{aligned} \tag{31}$$

Taking account of (18), (31) and integrating Eq. (29) over x from $-\Delta$ to Δ, where $\Delta \to 0$, we find the conditions for discontinuities in the derivatives of the probabilities $w_{1s}(x)$ and $w_{1c}(x)$ at the point $x = 0$:

$$\left.\frac{dw_{1s}}{dx}\right|_{x=0_+} - \left.\frac{dw_{1s}}{dx}\right|_{x=0_-} = -\frac{2(a_1+a_2)}{K}\, w_{1s}(0),$$
$$\left.\frac{dw_{1c}}{dx}\right|_{x=0_+} - \left.\frac{dw_{1c}}{dx}\right|_{x=0_-} = -\frac{2(a_1+a_2)}{K}\, w_{1c}(0). \tag{32}$$

Integrating Eq. (29) over x from $-x_2$ to x_1, we find the relationships between the derivatives of $w_{1s}(x)$ and $w_{1c}(x)$ at the points x_1 and $-x_2$:

$$\left.\frac{dw_{1s}}{dx}\right|_{x=x_1} - \left.\frac{dw_{1s}}{dx}\right|_{x=-x_2} = -\frac{2(a_1+a_2)}{K}\, w_{1s}(x_1),$$
$$\left.\frac{dw_{1c}}{dx}\right|_{x=x_1} - \left.\frac{dw_{1c}}{dx}\right|_{x=-x_2} = -\frac{2(a_1+a_2)}{K}\, w_{1c}(x_1). \tag{33}$$

So, to find $w(x,t)$, it is necessary to solve Eqs. (27), (29), (31) with boundary conditions (31), (32) and (33). For $f(x)$ described by (18), these equations can be solved exactly yielding the general solution

$$w_0(x) = \begin{cases} \left(C_0 + \dfrac{G_0}{a_1}\right)\exp\left(-\dfrac{2a_1 x}{K}\right) - \dfrac{G_0}{a_1} & \text{for } 0 < x < x_1, \\ \left(C_0 - \dfrac{G_0}{a_2}\right)\exp\left(\dfrac{2a_2 x}{K}\right) + \dfrac{G_0}{a_2} & \text{for } -x_2 < x < 0. \end{cases} \tag{34}$$

It follows from (31) that $G_0 = 0$ and

$$C_0 = 2\,\frac{a_1 a_2}{a_2 - a_1}\left[1 - \exp\left(-\frac{2U_0}{K}\right)\right]^{-1}, \tag{35}$$

where $U_0 = a_1 x_1 = a_2 x_2$ is the height of the potential barrier. The general solution of Eq. (29) is

$$w_{1s}(x) = \begin{cases} w_{1s}^{(1)}(x) & \text{for } 0 < x < x_1, \\ w_{1s}^{(2)}(x) & \text{for } -x_2 < x < 0, \end{cases}$$
$$w_{1c}(x) = \begin{cases} w_{1c}^{(1)}(x) & \text{for } 0 < x < x_1, \\ w_{1c}^{(2)}(x) & \text{for } -x_2 < x < 0, \end{cases} \tag{36}$$

where

$$
\begin{aligned}
w_{1s}^{(1)}(x) &= -2\mathrm{Im}[w_{11}(x)], \quad w_{1s}^{(2)}(x) = -2\mathrm{Im}[w_{12}(x)],\\
w_{1c}^{(1)}(x) &= 2\mathrm{Re}[w_{11}(x)], \quad w_{1c}^{(2)}(x) = 2\mathrm{Re}[w_{11}(x)],\\
w_{11}(x) &= C_{11}\exp(k_{11}x) + C_{21}\exp(k_{21}x) - \frac{C_0 a_1}{\omega K}\exp\left(-\frac{2a_1 x}{K}\right),\\
w_{12}(x) &= C_{12}\exp(k_{12}x) + C_{22}\exp(k_{22}x) + \frac{C_0 a_2}{\omega K}\exp\left(\frac{2a_2 x}{K}\right),\\
k_{11} &= -\frac{a_1 + \sqrt{a_1^2 + 2i\omega K}}{K}, \quad k_{21} = -\frac{a_1 - \sqrt{a_1^2 + 2i\omega K}}{K},\\
k_{12} &= \frac{a_2 - \sqrt{a_2^2 + 2i\omega K}}{K}, \quad k_{22} = \frac{a_2 + \sqrt{a_2^2 + 2i\omega K}}{K}.
\end{aligned}
\tag{37}
$$

The complex constants C_{11}, C_{21}, C_{12} and C_{22} can be found from the conditions (31), (32) and (33). They are proportional to C_0 and take the form

$$
\begin{aligned}
C_{11} &= F_{11}(a_1, a_2, r_1, r_2, U_0/K)C_0, \quad C_{12} = F_{12}(a_1, a_2, r_1, r_2, U_0/K)C_0,\\
C_{21} &= F_{21}(a_1, a_2, r_1, r_2, U_0/K)C_0, \quad C_{22} = F_{22}(a_1, a_2, r_1, r_2, U_0/K)C_0,
\end{aligned}
\tag{38}
$$

where $U_0 = a_1 x_1 = a_2 x_2$ is the height of the potential barrier, F_{11}, F_{21}, F_{12} and F_{22} are certain functions of their arguments.

It remains to find the general solution of Eq. (31). It is

$$
w_{20}(x) = \begin{cases} w_{20}^{(1)}(x) & \text{for } 0 < x < x_1,\\ w_{20}^{(2)}(x) & \text{for } -x_2 < x < 0, \end{cases}
\tag{39}
$$

where

$$
\begin{aligned}
w_{20}^{(1)}(x) &= Q_1 \exp\left(-\frac{2a_1 x}{K}\right) - \frac{G_2}{a_1}\\
&\quad -2\mathrm{Im}\left(\frac{C_{11}}{Kk_{11} + 2a_1}\exp(k_{11}x) + \frac{C_{21}}{Kk_{21} + 2a_1}\exp(k_{21}x)\right),\\
w_{20}^{(2)}(x) &= Q_2 \exp\left(\frac{2a_2 x}{K}\right) + \frac{G_2}{a_2}\\
&\quad -2\mathrm{Im}\left(\frac{C_{12}}{Kk_{12} - 2a_2}\exp(k_{12}x) + \frac{C_{22}}{Kk_{22} - 2a_2}\exp(k_{22}x)\right).
\end{aligned}
\tag{40}
$$

Here G_2 is the constant component of the probability flux in the second order approximation with respect to B.

From conditions (31) we obtain

$$\frac{G_2(a_1+a_2)}{a_1a_2} = Q_1 - Q_2 - 2\mathrm{Im}\left(\frac{C_{11}}{Kk_{11}+2a_1} + \frac{C_{21}}{Kk_{21}+2a_1} - \frac{C_{12}}{Kk_{12}-2a_2} - \frac{C_{22}}{Kk_{22}-2a_2}\right),$$

$$(Q_1-Q_2)\exp\left(-\frac{2U_0}{K}\right) - \frac{G_2}{a_1} - \frac{G_2}{a_2} - 2\mathrm{Im}\left[\frac{C_{11}\exp(k_{11}x_1)}{Kk_{11}+2a_1} + \frac{C_{21}\exp(k_{21}x_1)}{Kk_{21}+2a_1} - \frac{C_{12}\exp(-k_{12}x_2)}{Kk_{12}-2a_2} - \frac{C_{22}\exp(-k_{22}x_2)}{Kk_{22}-2a_2}\right] = 0,$$

$$\frac{K}{2}\left(\frac{Q_1}{a_1} + \frac{Q_2}{a_2}\right)\left[1-\exp\left(-\frac{2U_0}{K}\right)\right] + 2\mathrm{Im}\left[\frac{C_{11}(1-\exp(k_{11}x_1))}{(Kk_{11}+2a_1)k_{11}} + \frac{C_{21}(1-\exp(k_{21}x_1))}{(Kk_{21}+2a_1)k_{21}} - \frac{C_{12}(1-\exp(-k_{12}x_2))}{(Kk_{12}-2a_2)k_{12}} - \frac{C_{22}(1-\exp(-k_{22}x_2))}{(Kk_{22}-2a_2)k_{22}}\right] - G_2\left(\frac{x_1}{a_1} - \frac{x_2}{a_2}\right) = 0. \quad (41)$$

Solving Eqs (41) we find

$$G_2 = -\frac{2a_1a_2}{a_1+a_2}\left[1-\exp\left(-\frac{2U_0}{K}\right)\right]^{-1} \mathrm{Im}\left\{\frac{C_{11}}{Kk_{11}+2a_1}\left[\exp(k_{11}x_1) - \exp\left(-\frac{2U_0}{K}\right)\right] + \frac{C_{21}}{Kk_{21}+2a_1}\left[\exp(k_{21}x_1) - \exp\left(-\frac{2U_0}{K}\right)\right] - \frac{C_{12}}{Kk_{12}-2a_2}\left[\exp(-k_{12}x_2) - \exp\left(-\frac{2U_0}{K}\right)\right] - \frac{C_{22}}{Kk_{22}-2a_2}\left[\exp(-k_{22}x_2) - \exp\left(-\frac{2U_0}{K}\right)\right]\right\}. \quad (42)$$

Examples of the dependences of $V = \overline{v}/B^2 = G_2L$ on K/U_0 are shown in Fig. 4 for $a_1 = 1.25$, $a_2 = 5$, $x_1 = 0.8$, $x_2 = 0.2$ and different values of the frequency ω. Comparing these results with those found by use of quasi-stationary approximation[20] we see that they nearly coincide for $\omega \leq 0.5$. For most values of ω the mean particle velocity decreases and its maximum is shifted only weakly in the direction of larger values of K/U_0.

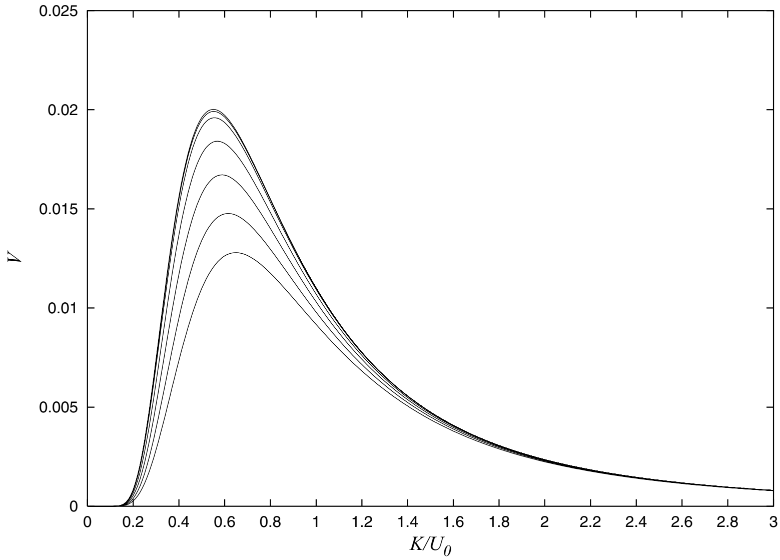

Fig. 4. The dependences of $V = \overline{v}/B^2 = G_2 L$ on K/U_0 for $a_1 = 1.25$, $a_2 = 5$, $U_0 = 1$ and (top-down) $\omega = 0.1,\ 0.5,\ 1,\ 2,\ 3,\ 4$ and 5.

4. STOCHASTIC RESONANCE

4.1. Stochastic Resonance in an Overdamped Oscillator

In stochastic resonance, a signal (usually a weak periodic one) in a nonlinear system can be optimally enhanced by the addition of noise of appropriate intensity. The phenomenon has been considered for a very wide range of physical situations,[36,37] including systems with either bistable or monostable potentials, undergoing motion that can be either underdamped or overdamped. Most often it has been considered in relation to the overdamped motion of a light particle in the simplest bistable potential field disturbed by a weak periodic signal and additive white noise:[18,38]

$$\dot{x} + x^3 - x = A\cos\omega t + \xi(t), \tag{43}$$

where x is the particle displacement, $A\cos\omega t$ is the weak periodic signal at frequency ω, $\xi(t)$ is white noise of intensity K, i.e. $\langle\xi(t)\xi(t+\tau) = K\delta(\tau)$. Use of this particular example (43) makes it difficult to identify the influence of the damping factor, however, so in the present work we will consider instead the system

$$2\delta\dot{x} + x^3 - x = A\cos\omega t + \xi(t), \tag{44}$$

where δ is the damping factor.

The treatment of stochastic resonance as a noise-induced change in the system's effective parameters was first considered by Landa.[20,21] We now generalise the main results of these works to be applicable to Eq. (44).

It follows from[39] that the power spectra of a solution of Eq. (44) contain both discrete frequency components (odd harmonics of the frequency ω) and continuous components caused by noise. We can therefore represent $x(t)$ in the form

$$x(t) = s(t) + n(t), \tag{45}$$

where

$$s(t) = \langle x(t) \rangle = \sum_{k=0}^{\infty} B_{2k+1} \cos((2k+1)\omega t + \psi_{2k+1}), \quad \langle n(t) \rangle = 0. \tag{46}$$

We will refer to the ratio of B_1 to A as the *gain factor* and denote it as $Q(K)$.

Further we substitute (45) into Eq. (44) and split it into two equations, of which one describes the quantities averaged over the statistical ensemble and the other describes the deviations from those averaged values. Since at $s = 0$ all odd moments of the noise $m_j = \langle n^j \rangle$ are equal to zero, in the first approximation with respect to s, we can set $m_3 = as + b\dot{s}$, where a and b are unknown functions of K and ω that will be found later. We therefore write the equations for $s(t)$ and $n(t)$ as

$$(2\delta + b)\dot{s} + cs + s^3 = A\cos\omega t, \tag{47}$$

$$2\delta\dot{n} + (3s^2 - 1)n + n^3 + (3n^2 - 1 - c)s - b\dot{s} = \xi(t), \tag{48}$$

where

$$c = 3m_2 - 1 + a \tag{49}$$

is the effective stiffness, and b is the addition to the damping factor caused by noise.

We emphasize that such a splitting of the initial equation into two equations is similar to the separation of motions into slow and fast ones suggested in ref. 22 and used in refs. 40, 41 for calculations of vibrational resonance. The difference here is that we separate the motions, not into slow and fast, but into regular (averaged) and random parts.

To calculate the moments m_j and find a and b, we use the Fokker–Planck equation corresponding to the Langevin equation (48). In a linear approximation with respect to $s(t)$ it is

$$2\delta \frac{\partial w}{\partial t} = \frac{\partial}{\partial n}[(n^3 - n + (3n^2 - 1 - c)s - b\dot{s})\, w] + \frac{K}{4\delta} \frac{\partial^2 w}{\partial n^2}\,. \tag{50}$$

It is convenient to seek a solution of Eq. (50) in the form of a sum of three components, of which w_0 is the main one; the other two, w_1 and w_2, are small in comparison with w_0. Introducing a conventional small parameter ϵ and assuming $s \sim \dot{s} \sim \epsilon$, we set

$$w(n,t) = w_0(n) + \epsilon\left(w_1(n)s + \frac{w_2(n)}{\omega}\,\dot{s}\right). \tag{51}$$

Because in the linear approximation $s(t)$ is a harmonic signal of frequency ω, we have $\ddot{s}(t) = -\omega^2 s(t)$. Taking this into account, substituting (51) into Eq. (50) and restricting ourselves to terms of the first order with respect to ϵ, we obtain the following equations for the components in question:

$$\frac{K}{4\delta}\frac{\partial w_0}{\partial n} + (n^3 - n)w_0 = 0, \tag{52}$$

$$\begin{aligned} &2\delta\omega w_1 - \frac{d}{dn}\left((n^3 - n)w_2 + \frac{K}{4\delta}\frac{dw_2}{dn}\right) = -\omega b\,\frac{dw_0}{dn}\,, \\ &2\delta\omega w_2 + \frac{d}{dn}\left((n^3 - n)w_1 + \frac{K}{4\delta}\frac{dw_1}{dn}\right) = -\frac{d}{dn}[(3n^2 - 1 - c)w_0]. \end{aligned} \tag{53}$$

It can be seen from Eqs. (52), (53) that $w_0(n)$ is an even function of n, whereas we can seek $w_1(n)$ and $w_2(n)$ as odd functions of n. In so doing we have

$$\int_{-\infty}^{\infty} w_{1,2}(n)\,dn = 0, \tag{54}$$

$$\int_{-\infty}^{\infty} n^2 w_0(n)\,dn = m_2. \tag{55}$$

It follows from (54) that the normalization condition for the probability $w(n,t)$ becomes

$$\int_{-\infty}^{\infty} w_0(n)\,dn = 1. \tag{56}$$

The conditions $\langle n\rangle = 0$ and $\langle n^3\rangle = as + b\dot{s}$ lead to

$$\begin{aligned} &\int_{-\infty}^{\infty} n w_1(n)\,dn = 0, \quad \int_{-\infty}^{\infty} n w_2(n)\,dn = 0, \\ &\int_{-\infty}^{\infty} n^3 w_1(n)\,dn = a, \quad \int_{-\infty}^{\infty} n^3 w_2(n)\,dn = \omega b. \end{aligned} \tag{57}$$

Solving Eq. (52) we find $w_0(n)$:

$$w_0(n) = C \exp\left(-\frac{\delta(n^4 - 2n^2)}{K}\right), \tag{58}$$

where C is the normalization factor. Substituting (58) into (55) we find m_2 as a function of K and δ.

An odd solution of Eq. (53) can be written as

$$\begin{aligned} w_1(n) &= C_1 w_{11}(n) + C_2 w_{12}(n) + w_{10}(n) + w_{1c}(n)c + w_{1b}(n)\omega b, \\ w_2(n) &= C_1 w_{21}(n) + C_2 w_{22}(n) + w_{20}(n) + w_{2c}(n)c + w_{2b}(n)\omega b, \end{aligned} \tag{59}$$

where $w_{11}(n)$, $w_{21}(n)$ and $w_{12}(n)$, $w_{22}(n)$ are two fundamental partial solutions of homogeneous Eq. (55) satisfying the conditions

$$\begin{aligned} &w_{11}(0) = 0, \quad w_{21}(0) = 0, \quad \frac{dw_{11}(0)}{dn} = 1, \quad \frac{dw_{21}(0)}{dn} = 0, \\ &w_{12}(0) = 0, \quad w_{22}(0) = 0, \quad \frac{dw_{12}(0)}{dn} = 0, \quad \frac{dw_{22}(0)}{dn} = 1, \end{aligned} \tag{60}$$

C_1 and C_2 are arbitrary constants, and $w_{10}(n)$, $w_{1c}(n)$, $w_{1b}(n)$, $w_{20}(n)$, $w_{2b}(n)$ and $w_{2c}(n)$ are described by the equations

$$\begin{aligned} &2\delta\omega w_{10} - \frac{d}{dn}\left((n^3 - n)w_{20} + \frac{K}{4\delta}\frac{dw_{20}}{dn}\right) = 0, \\ &2\delta\omega w_{20} + \frac{d}{dn}\left((n^3 - n)w_{10} + \frac{K}{4\delta}\frac{dw_{10}}{dn}\right) = n\left(\frac{4(3n^2-1)(n^2-1)\delta}{K} - 6\right)w_0(n), \\ &2\delta\omega w_{1c} - \frac{d}{dn}\left((n^3 - n)w_{2c} + \frac{K}{4\delta}\frac{dw_{2c}}{dn}\right) = 0, \\ &2\delta\omega w_{2c} + \frac{d}{dn}\left((n^3 - n)w_{1c} + \frac{K}{4\delta}\frac{dw_{1c}}{dn}\right) = -\frac{4n(n^2-1)\delta}{K}w_0(n), \\ &2\delta\omega w_{1b} - \frac{d}{dn}\left((n^3 - n)w_{2b} + \frac{K}{4\delta}\frac{dw_{2b}}{dn}\right) = \frac{4n(n^2-1)\delta}{K}w_0(n), \\ &2\delta\omega w_{2b} + \frac{d}{dn}\left((n^3 - n)w_{1b} + \frac{K}{4\delta}\frac{dw_{1b}}{dn}\right) = 0. \end{aligned} \tag{61}$$

Equation (61) should be solved with zero initial conditions.

The equations for the unknown parameters a, b, c, and arbitrary constants C_1 and C_2 follow from (57), (59) and (49):

$$\begin{aligned} J_{11}C_1 + J_{12}C_2 + J_{10} + cJ_{1c} + \omega bJ_{1b} &= 0, \\ J_{21}C_1 + J_{22}C_2 + J_{20} + cJ_{2c} + \omega bJ_{2b} &= 0, \end{aligned} \tag{62}$$

$$I_{11}C_1 + I_{12}C_2 + I_{10} + cI_{1c} + \omega b I_{1b} = c + 1 - 3m_2,$$
$$I_{21}C_1 + I_{22}C_2 + I_{20} + cI_{2c} + \omega b I_{2b} = \omega b,$$

where

$$J_{ij} = \int_{-\infty}^{\infty} n w_{ij}(n)\, dn, \quad I_{ij} = \int_{-\infty}^{\infty} n^3 w_{ij}(n)\, dn.$$

These equations allow us to find the effective stiffness c, and the addition to damping factor b, as functions of K and δ.

The dependences of c, ωb, and a on the noise intensity K for $\delta = 0.5$ and different values of the signal frequency ω are shown in Fig. 5. To clarify the effect of the damping factor we construct the same dependences for $\omega = 0.1$ and three values of δ ($\delta = 0.25$, $\delta = 0.5$ and $\delta = 1$), see Fig. 6. It is seen that with increasing δ all curves are displaced to larger values of the noise intensity and the maximal value of the correction to the damping factor slightly decreases.

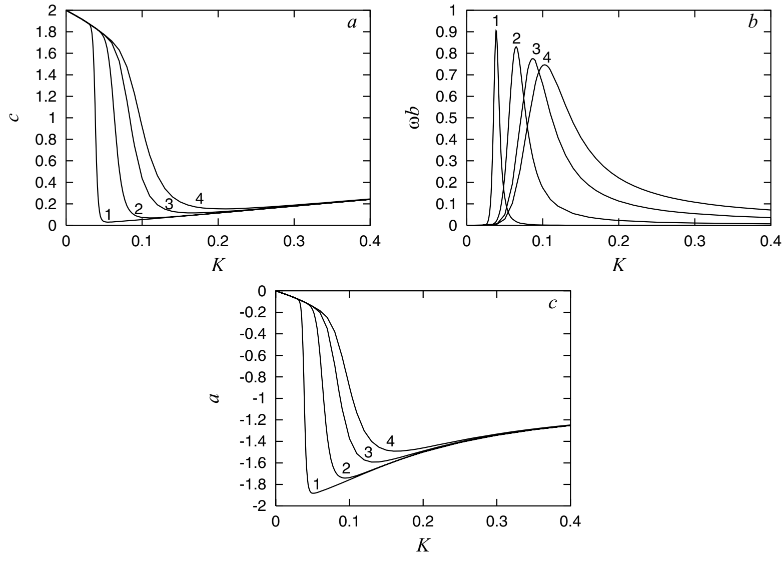

Fig. 5. The dependences of c, ωb and a on the noise intensity K for $\delta = 0.5$ and $\omega = 0.0001$ (curves 1), $\omega = 0.01$ (curves 2), $\omega = 0.05$ (curves 3) and $\omega = 0.1$ (curves 4).

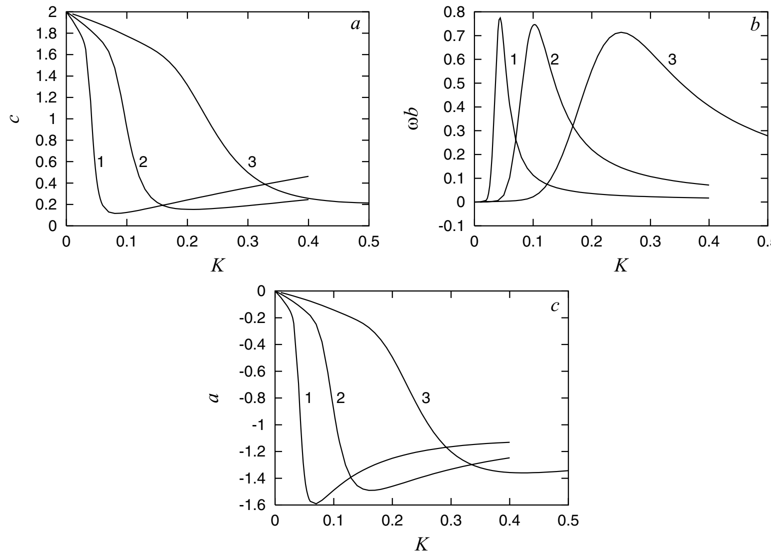

Fig. 6. The dependences of c, ωb and a on the noise intensity K for $\omega = 0.1$ and $\delta = 0.25$ (curves 1) and $\delta = 0.5$ (curves 2).

4.2. Stochastic Resonance in the Case of a Particle of Small Mass Moving in the Simplest Bistable Potential Field

We consider the underdamped motion of a particle described by the equation

$$\mu^2 \ddot{x} + 2\delta \dot{x} + x^3 - x = A \cos \omega t + \xi(t), \tag{63}$$

where μ^2 is a small parameter proportional to the particle mass.

As before, we represent the solution of Eq. (63) as a sum of the signal $s(t) = \langle x(t) \rangle$ and noise $n(t)$. It is evident that the mean value of noise must be equal to zero, whereas the third moment can differ from zero in the presence of a signal. In the linear approximation with respect to $s(t)$ we obtain the following equations for $s(t)$ and $n(t)$:

$$\mu^2 \ddot{s} + (2\delta + b)\dot{s} + cs = A \cos \omega t, \tag{64}$$

$$\mu^2 \ddot{n} + 2\delta \dot{n} + n^3 - n + (3n^2 - 1 - c)s - b\dot{s} = \xi(t), \tag{65}$$

where the effective stiffness c is described by (48).

An equation of the same kind as (65), but with coefficients independent of time and with $2\delta = 1$, was considered by Stratonovich.[39] By analogy with,[39] Eq. (65) may be conveniently rewritten in the form of two following Langevin equations:

$$\mu \dot{n} = y, \quad \mu \dot{y} = -\frac{2\delta}{\mu} y - n^3 + n - (3n^2 - 1 - c)s + b\dot{s} + \xi(t). \tag{66}$$

The two-dimensional Fokker–Planck equation corresponding to Eq. (66) is

$$\begin{aligned} \frac{\partial w}{\partial t} = &- \frac{1}{\mu}\left(y \frac{\partial w}{\partial n} + [n - n^3 - (3n^2 - 1 - c)s + b\dot{s}]\frac{\partial w}{\partial y}\right) \\ &+ \frac{1}{\mu^2}\left(2\delta \frac{\partial (yw)}{\partial y} + \frac{K}{2}\frac{\partial^2 w}{\partial y^2}\right). \end{aligned} \tag{67}$$

Representing the solution of Eq. (67) as a sum of three components

$$w(n, y, t) = w_0(n, y) + w_1(n, y)s(t) + \frac{w_2(n, y)\dot{s}(t)}{\omega}, \tag{68}$$

for $w_0(n, y)$, $w_1(n, y)$ and $w_2(n, y)$ we obtain the following equations:

$$\left(2\delta \frac{\partial (yw_0)}{\partial y} + \frac{K}{2}\frac{\partial^2 w_0}{\partial y^2}\right) - \mu\left(y \frac{\partial w_0}{\partial n} - (n^3 - n)\frac{\partial w_0}{\partial y}\right) = 0, \tag{69}$$

$$\begin{aligned} &2\delta \frac{\partial (yw_1)}{\partial y} + \frac{K}{2}\frac{\partial^2 w_1}{\partial y^2} - \mu\left(y \frac{\partial w_1}{\partial n} - (n^3 - n)\frac{\partial w_1}{\partial y} - \mu\omega w_2\right) \\ &\quad = -\mu(3n^2 - 1 - c)\frac{dw_0}{dy}, \\ &2\delta \frac{\partial (yw_2)}{\partial y} + \frac{K}{2}\frac{\partial^2 w_2}{\partial y^2} - \mu\left(y \frac{\partial w_2}{\partial n} - (n^3 - n)\frac{\partial w_2}{\partial y} + \mu\omega w_1\right) = \mu\omega b \frac{dw_0}{dy}. \end{aligned} \tag{70}$$

As shown in ref. 39, a solution of Eq. (69) can be found by setting to zero each of the terms in parentheses. We thus find

$$w_0(n, y) = C_0 \exp\left[-\frac{4\delta}{K}\left(\frac{y^2}{2} + u(n)\right)\right], \tag{71}$$

where $u(n) = n^4/4 - n^2/2$, and C_0 is the normalization constant. It should be emphasized that $w_0(n, y)$ is independent of the parameter μ.

To find the unknown parameters c and b, it is necessary to calculate the probability distributions

$$v_{1,2}(n) = \int_{-\infty}^{\infty} w_{1,2}(n, y)\, dy. \tag{72}$$

To do this, we solve Eq. (70) by expansion in eigenfunctions $Y_m(y)$ of the boundary value problem described by the equation

$$\frac{K}{4\delta}\frac{d^2Y}{dy^2} + \frac{d(yY)}{dy} + \lambda Y = 0 \tag{73}$$

with boundary conditions $Y(\pm\infty) = 0$. We note that each of Eq. (70) for $\mu = 0$ reduces to Eq. (73) with $\lambda = 0$.

As discussed earlier in refs. 39 and 20, the eigenvalues $\lambda_m = m$, and the eigenfunctions $Y_m(y)$ can be expressed in terms of Hermite polynomials $H_m(z)$ as

$$Y_m(y) = (-1)^m\sqrt{\frac{2\delta}{\pi K 2^m m!}}\,\exp\left(-\frac{2\delta y^2}{K}\right) H_m\left(y\sqrt{\frac{2\delta}{K}}\right). \tag{74}$$

The functions $Y_m(y)$ must satisfy the following conditions of orthogonality and normalization:

$$\int_{-\infty}^{\infty}\frac{Y_m(y)Y_l(y)}{Y_0(y)}\,dy = \delta_{ml}, \tag{75}$$

where δ_{ml} is the Kronecker delta. In addition, the following relationships follow from the properties of Hermite polynomials:[20]

$$\int_{-\infty}^{\infty} Y_m(y)\,dy = \delta_{m0}, \tag{76}$$

$$\begin{aligned}
\frac{dY_m(y)}{dy} &= \sqrt{\frac{4\delta(m+1)}{K}}\,Y_{m+1}(y),\\
yY_m(y) &= -\sqrt{\frac{K}{4\delta}}(\sqrt{m+1}\,Y_{m+1}(y) + \sqrt{m}\,Y_{m-1}(y)),\\
\frac{d(yY_m(y))}{dy} &= -(\sqrt{(m+1)(m+2)}\,Y_{m+2}(y) + mY_m(y)).
\end{aligned} \tag{77}$$

The expansion of the solution of Eq. (70) in terms of $Y_m(y)$ can be written as

$$w_1(n, y) = \sum_{m=0}^{\infty} v_{1m}(n)Y_m(y), \quad w_2(n, y) = \sum_{m=0}^{\infty} v_{2m}(n)Y_m(y). \tag{78}$$

Integrating (78) over y and using property (76) we find

$$v_1(n) = v_{10}(n), \quad v_2(n) = v_{20}(n). \tag{79}$$

It follows from here that for solution of our problem it is sufficiently to find the functions $v_{10}(n)$ and $v_{20}(n)$.

Substituting (78) into Eq. (70) and taking into account that

$$w_0(n) = v_0(n)Y_0(y),$$

where

$$v_0(n) = C\exp\left(-\frac{4\delta u(n)}{K}\right), \quad C = \sqrt{\frac{\pi\mu K}{2\delta}}\,C_0,$$

we obtain

$$\begin{aligned}
&\sum_{m=0}^{\infty}\Bigg(\sqrt{\frac{K}{4\delta}}\,(\sqrt{m+1}\,Y_{m+1}(y)+\sqrt{m}\,Y_{m-1}(y))\frac{dv_{1m}}{dn}\\
&\quad+\sqrt{\frac{4\delta(m+1)}{K}}\,(n^3-n)Y_{m+1}(y)v_{1m}(n)\\
&\quad-\frac{2\delta}{\mu}\,mY_m(y)v_{1m}(n)+\mu\omega Y_m(y)v_{2m}(n)\Bigg)\\
&=-\sqrt{\frac{4\delta}{K}}\,(3n^2-1-c)Y_1(y)v_0(n),\\
&\sum_{m=0}^{\infty}\Bigg(\sqrt{\frac{K}{4\delta}}\,(\sqrt{m+1}\,Y_{m+1}(y)+\sqrt{m}\,Y_{m-1}(y))\frac{dv_{2m}}{dn}\\
&\quad+(n^3-n)\sqrt{\frac{4\delta(m+1)}{K}}\,Y_{m+1}(y)v_{2m}(n)\Bigg)\\
&\quad-\frac{2\delta}{\mu}\,mY_m(y)v_{2m}(n)-\mu\omega Y_m(y)v_{1m}(n)\\
&=\sqrt{\frac{4\delta}{K}}\,\omega bY_1(y)v_0(n).
\end{aligned} \tag{80}$$

Equating the terms of $Y_m(y)$ that have the same subscripts and setting $m = 0, 1, 2, \ldots$ we obtain the following equations:

$$\sqrt{\frac{K}{4\delta}}\,\frac{dv_{11}}{dn}+\mu\omega v_{20}=0, \quad \sqrt{\frac{K}{4\delta}}\,\frac{dv_{21}}{dn}-\mu\omega v_{10}=0,$$

$$\begin{aligned}
&\sqrt{\frac{K}{4\delta}}\left(\frac{dv_{10}}{dn}+\sqrt{2}\,\frac{dv_{12}}{dn}\right)+\sqrt{\frac{4\delta}{K}}\,(n^3-n)\,v_{10}-\frac{2\delta}{\mu}\,v_{11}\\
&\qquad+\mu\omega v_{21}=-\sqrt{\frac{4\delta}{K}}\,(3n^2-1-c)v_0(n),
\end{aligned} \tag{81}$$

$$\sqrt{\frac{K}{4\delta}}\left(\frac{dv_{20}}{dn}+\sqrt{2}\,\frac{dv_{22}}{dn}\right)+\sqrt{\frac{4\delta}{K}}\,(n^3-n)\,v_{20}-\frac{2\delta}{\mu}\,v_{21}$$
$$-\mu\omega v_{11}=\sqrt{\frac{4\delta}{K}}\,\omega b v_0(n),$$

$$\sqrt{\frac{K}{4\delta}}\left(\sqrt{j}\,\frac{dv_{1,j-1}}{dn}+\sqrt{j+1}\,\frac{dv_{1,j+1}}{dn}\right)$$
$$+\sqrt{\frac{4j\delta}{K}}\,(n^3-n)v_{1,j-1}-\frac{2j\delta}{\mu}\,v_{1j}+\mu\omega v_{2j}=0\qquad(j\geq 2),\quad(82)$$
$$\sqrt{\frac{K}{4\delta}}\left(\sqrt{j}\,\frac{dv_{2,j-1}}{dn}+\sqrt{j+1}\,\frac{dv_{2,j+1}}{dn}\right)$$
$$+\sqrt{\frac{4j\delta}{K}}\,(n^3-n)v_{2,j-1}-\frac{2j\delta}{\mu}\,v_{2j}-\mu\omega v_{1j}=0\qquad(j\geq 2).$$

Expanding v_{1m} and v_{2m} as power series in μ

$$v_{1m}=\sum_{k=0}^{\infty}\mu^k v_{1m}^{(k)},\quad v_{2m}=\sum_{k=0}^{\infty}\mu^k v_{2m}^{(k)}.\qquad(83)$$

Equating the coefficients of the same powers of μ and taking into account that

$$v_{10}^{(2k+1)}=0,\quad v_{20}^{(2k+1)}=0,\quad v_{11}^{(2k)}=0,\quad v_{21}^{(2k)}=0,\quad v_{1j}^{(l)}=v_{2j}^{(l)}=0\quad(j\geq 2l),$$

we find from Eq. (81)

– For μ^0:

$$\omega v_{10}^{(0)}-\sqrt{\frac{K}{4\delta}}\,\frac{dv_{21}^{(1)}}{dn}=0,\quad \omega v_{20}^{(0)}+\sqrt{\frac{K}{4\delta}}\,\frac{dv_{11}^{(1)}}{dn}=0,$$

$$v_{11}^{(1)}=\frac{1}{2\delta}\left(\sqrt{\frac{K}{4\delta}}\,\frac{dv_{10}^{(0)}}{dn}+\sqrt{\frac{4\delta}{K}}\,(n^3-n)v_{10}^{(0)}+\sqrt{\frac{4\delta}{K}}\,(3n^2-1-c)v_0(n)\right),\quad(84)$$

$$v_{21}^{(1)}=\frac{1}{2\delta}\left(\sqrt{\frac{K}{4\delta}}\,\frac{dv_{20}^{(0)}}{dn}+\sqrt{\frac{4\delta}{K}}\,(n^3-n)v_{20}^{(0)}-\sqrt{\frac{4\delta}{K}}\,\omega b v_0(n)\right);$$

– For μ^2:

$$
\begin{aligned}
&\omega v_{10}^{(2)} - \sqrt{\frac{K}{4\delta}}\,\frac{dv_{21}^{(3)}}{dn} = 0, \quad \omega v_{20}^{(2)} + \sqrt{\frac{K}{4\delta}}\,\frac{dv_{11}^{(3)}}{dn} = 0,\\
&v_{11}^{(3)} = \frac{1}{2\delta}\left[\sqrt{\frac{K}{4\delta}}\left(\frac{dv_{10}^{(2)}}{dn} + \frac{4\delta}{K}(n^3-n)v_{10}^{(2)} + \sqrt{2}\,\frac{dv_{12}^{(2)}}{dn}\right) + \omega v_{21}^{(1)}\right], \qquad (85)\\
&v_{21}^{(3)} = \frac{1}{2\delta}\left[\sqrt{\frac{K}{4\delta}}\left(\frac{dv_{20}^{(2)}}{dn} + \frac{4\delta}{K}(n^3-n)v_{20}^{(2)} + \sqrt{2}\,\frac{dv_{22}^{(2)}}{dn}\right) - \omega v_{11}^{(1)}\right],
\end{aligned}
$$

$$
\begin{aligned}
&v_{12}^{(2)} = \frac{\sqrt{2}}{4\delta}\left(\sqrt{\frac{4\delta}{K}}(n^3-n)v_{11}^{(1)} - \omega v_{20}^{(0)}\right),\\
&v_{22}^{(2)} = \frac{\sqrt{2}}{4\delta}\left(\sqrt{\frac{4\delta}{K}}(n^3-n)v_{21}^{(1)} + \omega v_{10}^{(0)}\right);
\end{aligned}
\qquad (86)
$$

– For μ^4:

$$
\begin{aligned}
&\omega v_{10}^{(4)} - \sqrt{\frac{K}{4\delta}}\,\frac{dv_{21}^{(5)}}{dn} = 0, \quad \omega v_{20}^{(4)} + \sqrt{\frac{K}{4\delta}}\,\frac{dv_{11}^{(5)}}{dn} = 0,\\
&v_{11}^{(5)} = \frac{1}{2\delta}\left[\sqrt{\frac{K}{4\delta}}\left(\frac{dv_{10}^{(4)}}{dn} + \frac{4\delta}{K}(n^3-n)v_{10}^{(4)} + \sqrt{2}\,\frac{dv_{12}^{(4)}}{dn}\right) + \omega v_{21}^{(3)}\right], \qquad (87)\\
&v_{21}^{(5)} = \frac{1}{2\delta}\left[\sqrt{\frac{K}{4\delta}}\left(\frac{dv_{20}^{(4)}}{dn} + \frac{4\delta}{K}(n^3-n)v_{20}^{(4)} + \sqrt{2}\,\frac{dv_{22}^{(4)}}{dn}\right) - \omega v_{11}^{(3)}\right],
\end{aligned}
$$

$$
\begin{aligned}
&v_{12}^{(4)} = \frac{1}{4\delta}\left[\sqrt{\frac{K}{2\delta}}\left(\frac{dv_{11}^{(3)}}{dn} + \frac{4\delta}{K}(n^3-n)v_{11}^{(3)} + \sqrt{\frac{3}{2}}\,\frac{dv_{13}^{(3)}}{dn}\right) + \omega v_{22}^{(2)}\right],\\
&v_{22}^{(4)} = \frac{1}{4\delta}\left[\sqrt{\frac{K}{2\delta}}\left(\frac{dv_{21}^{(3)}}{dn} + \frac{4\delta}{K}(n^3-n)v_{21}^{(3)} + \sqrt{\frac{3}{2}}\,\frac{dv_{23}^{(3)}}{dn}\right) - \omega v_{12}^{(2)}\right],\\
&v_{13}^{(3)} = \frac{1}{6\delta}\left[\sqrt{\frac{3K}{4\delta}}\left(\frac{dv_{12}^{(2)}}{dn} + \frac{4\delta}{K}(n^3-n)v_{12}^{(2)} + \sqrt{\frac{4}{3}}\,\frac{dv_{14}^{(2)}}{dn}\right) + \omega v_{23}^{(1)}\right],\\
&v_{23}^{(3)} = \frac{1}{6\delta}\left[\sqrt{\frac{3K}{4\delta}}\left(\frac{dv_{12}^{(2)}}{dn} + \frac{4\delta}{K}(n^3-n)v_{22}^{(2)} + \sqrt{\frac{4}{3}}\,\frac{dv_{24}^{(2)}}{dn}\right) - \omega v_{13}^{(1)}\right],
\end{aligned}
\qquad (88)
$$

Equations (84)–(87) can conveniently be reduced to the following:

$$\frac{d^2 v_{10}^{(0)}}{dn^2} + \frac{4\delta}{K}\left((n^3 - n)\frac{dv_{10}^{(0)}}{dn} + (3n^2 - 1)v_{10}^{(0)} + 2\omega\delta v_{20}^{(0)}\right)$$
$$= -\frac{4\delta}{K}\frac{d}{dn}\Big((3n^2 - 1 - c)\, v_0(n)\Big), \tag{89}$$
$$\frac{d^2 v_{20}^{(0)}}{dn^2} + \frac{4\delta}{K}\left((n^3 - n)\frac{dv_{20}^{(0)}}{dn} + (3n^2 - 1)v_{20}^{(0)} - 2\omega\delta v_{10}^{(0)}\right)$$
$$= \frac{4\delta}{K}\,\omega b\,\frac{dv_0(n)}{dn},$$

$$\frac{d^2 v_{10}^{(2)}}{dn^2} + \frac{4\delta}{K}\left((n^3 - n)\frac{dv_{10}^{(2)}}{dn} + (3n^2 - 1)v_{10}^{(2)} + 2\omega\delta v_{20}^{(2)}\right)$$
$$= -\left(\sqrt{2}\,\frac{d^2 v_{12}^{(2)}}{dn^2} + \omega^2\,\frac{4\delta}{K}\,v_{10}^{(0)}\right), \tag{90}$$
$$\frac{d^2 v_{20}^{(2)}}{dn^2} + \frac{4\delta}{K}\left((n^3 - n)\frac{dv_{20}^{(1)}}{dn} + (3n^2 - 1)v_{20}^{(1)} - 2\omega\delta v_{10}^{(1)}\right)$$
$$= -\left(\sqrt{2}\,\frac{d^2 v_{22}^{(2)}}{dn^2} + \omega^2\,\frac{4\delta}{K}\,v_{20}^{(0)}\right),$$

$$\frac{d^2 v_{10}^{(4)}}{dn^2} + \frac{4\delta}{K}\left((n^3 - n)\frac{dv_{10}^{(4)}}{dn} + (3n^2 - 1)v_{10}^{(4)} + 2\omega\delta v_{20}^{(4)}\right)$$
$$= -\left(\sqrt{2}\,\frac{d^2 v_{12}^{(4)}}{dn^2} + \omega\sqrt{\frac{4\delta}{K}}\,\frac{d2v_{21}^{(3)}}{dn}\right), \tag{91}$$
$$\frac{d^2 v_{20}^{(4)}}{dn^2} + \frac{4\delta}{K}\left((n^3 - n)\frac{dv_{20}^{(4)}}{dn} + (3n^2 - 1)v_{20}^{(4)} - 2\omega\delta v_{10}^{(4)}\right)$$
$$= -\left(\sqrt{2}\,\frac{d^2 v_{22}^{(4)}}{dn^2} - \omega\sqrt{\frac{4\delta}{K}}\,\frac{d2v_{11}^{(3)}}{dn}\right),$$

where $v_{12}^{(2)}$, $v_{22}^{(2)}$, $v_{12}^{(4)}$ and $v_{22}^{(4)}$ are described by Eqs. (86), (88).

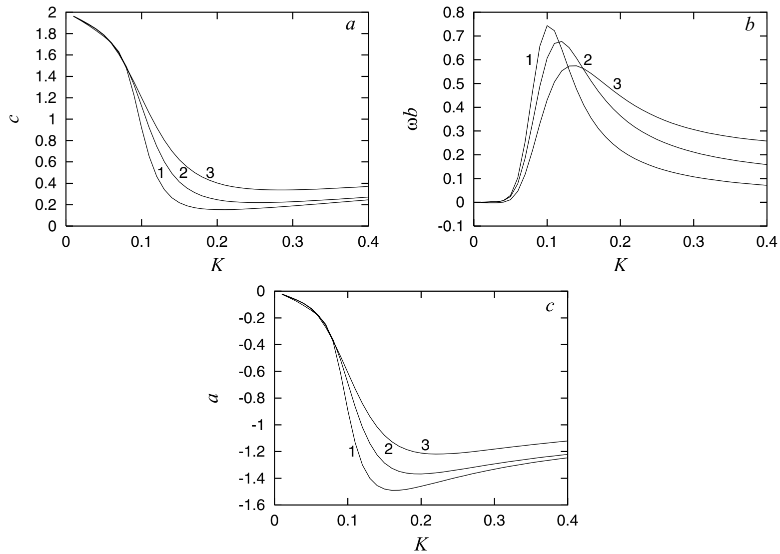

Fig. 7. The dependences of c, ωb and a on the noise intensity K for $\omega = 0.1$, $\delta = 0.5$ and $\mu = 0$ (curves 1), $\mu = 0.5$ (curves 2) and $\mu = 1$ (curves 3).

The conditions $\langle n \rangle = 0$ and $\langle n^3 \rangle = as + b\dot{s}$ are

$$\begin{aligned} &\int_{-\infty}^{\infty} n v_{10}(n)\, dn = 0, \qquad \int_{-\infty}^{\infty} n v_{20}(n)\, dn = 0, \\ &\int_{-\infty}^{\infty} n^3 v_{10}(n)\, dn = a, \qquad \int_{-\infty}^{\infty} n^3 v_{20}(n)\, dn = \omega b. \end{aligned} \tag{92}$$

These conditions, along with the normalization condition, allow us to find the effective stiffness c and the addition to damping factor ωb as functions of K, δ and μ.

The dependences of c, ωb and a on the noise intensity K for $\delta = 0.5, \omega = 0.1$ and $\mu = 0$, $\mu = 0.5$ and $\mu = 1$ are shown in Fig. 7. It can be seen that these parameters depend on the particle mass μ relatively weakly. Nevertheless, with increasing μ all curves are displaced towards larger values of the noise intensity, the minimum value of c increases, and the maximum value of ωb decreases.

5. CONCLUSION

By consideration of several physical examples we have shown that noise may cause changes in the effective parameters of averaged motion in nonlinear systems. This change found can be construed as the origin of apparent resonance phenomena in a system with only a half-degree-of-freedom, as seen in an over-damped oscillator, in the directed motion of a particle in the absence of a directed force, in the impossibility of Maxwell's demon, in the motion of a barge or a raft at a rate exceeding the stream velocity, and in many other interesting phenomena. We emphasize that each of the problems considered can be, and in most cases has been, analysed earlier using different theoretical techniques. These other approaches do not, however, allow us to find the change of the effective parameters of averaged motion which explain the physical mechanisms of the phenomena under consideration. We can reasonably speculate that the present approach, in terms of noise-induced parameter changes, will be of wide applicability enabling quantitative analyses of a diverse range of stochastic phenomena to be encompassed within a single, unified, conceptual framework.

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